

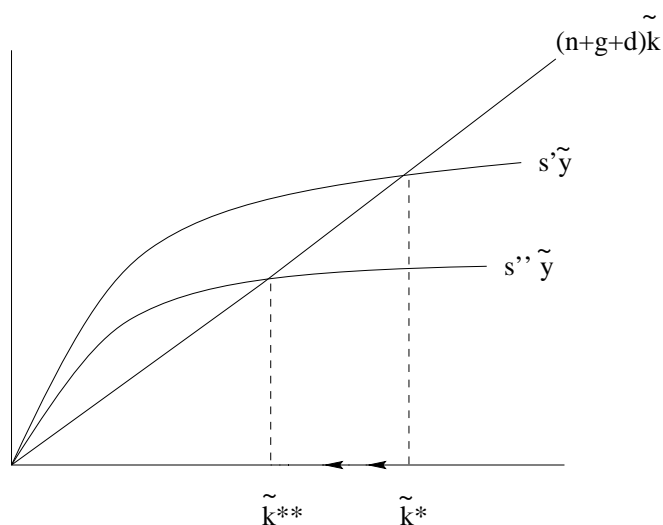
## 2 The Solow Model

**Exercise 1.** *A decrease in the investment rate.*

A decrease in the investment rate causes the  $s\tilde{y}$  curve to shift down: at any given level of  $\tilde{k}$ , the investment-technology ratio is lower at the new rate of saving/investment.

Assuming the economy began in steady state, the capital-technology ratio is now higher than is consistent with the reduced saving rate, so it declines gradually, as shown in Figure 1.

Figure 1: A Decrease in the Investment Rate



The log of output per worker  $y$  evolves as in Figure 2, and the dynamics of the growth rate are shown in Figure 3. Recall that  $\log \tilde{y} = \alpha \log \tilde{k}$  and  $\dot{\tilde{k}}/\tilde{k} = s''\tilde{k}^{\alpha-1} - (n + g + d)$ .

The policy permanently reduces the level of output per worker, but the growth rate per worker is only temporarily reduced and will return to  $g$  in the long run.

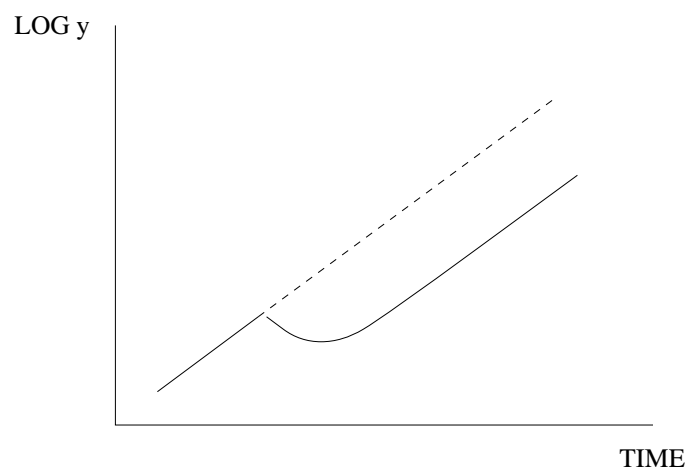
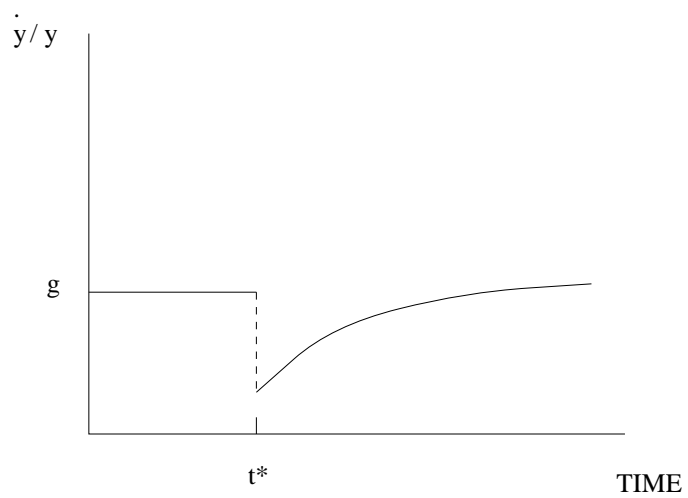
Figure 2:  $y(t)$ 

Figure 3: Growth Rate of Output per Worker

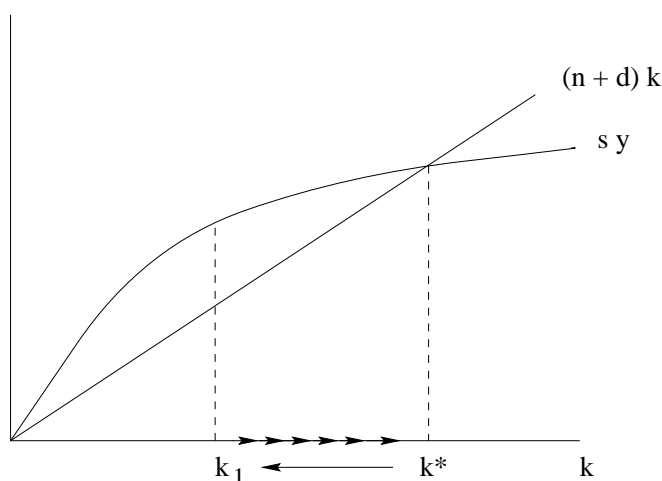


**Exercise 2.** *An increase in the labor force.*

The key to this question is to recognize that the initial effect of a sudden increase in the labor force is to reduce the capital-labor ratio since  $k \equiv K/L$  and  $K$  is fixed at a moment in time. Assuming the economy was in steady state prior to the increase in labor force,  $k$  falls from  $k^*$  to some new level  $k_1$ . Notice that this is a movement *along* the  $sy$  and  $(n+d)k$  curves rather than a *shift* of either schedule: both curves are plotted as functions of  $k$ , so that a change in  $k$  is a movement along the curves. (For this reason, it is somewhat tricky to put this question first!)

At  $k_1$ ,  $sy > (n+d)k_1$ , so that  $\dot{k} > 0$ , and the economy evolves according to the usual Solow dynamics, as shown in Figure 4.

Figure 4: An Increase in the Labor Force

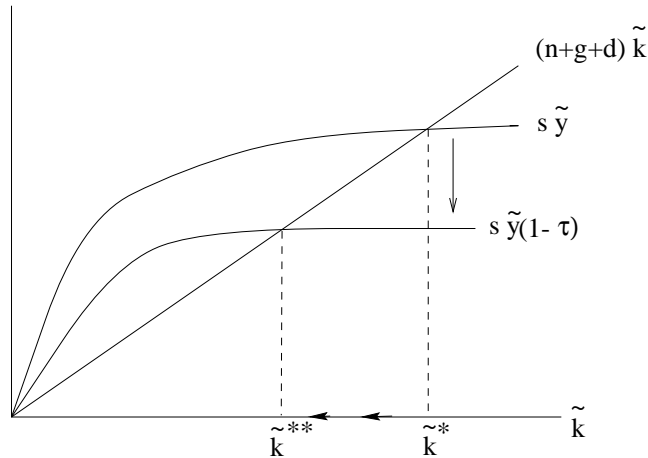


In the short run, per capita output and capital drop in response to a inlarge flow of workers. Then these two variables start to grow (at a decreasing rate), until in the long run per capita capital returns to the original level,  $k^*$ . In the long run, nothing has changed!

**Exercise 3.** *An income tax.*

Assume that the government throws away the resources it receives in taxes. Then an income tax reduces the total amount available for investing and shifts the investment curve down as shown in Figure 5.

Figure 5: An Income Tax



The tax policy permanently reduces the level of output per worker, but the growth rate per worker is only temporarily lowered. Notice that this experiment has basically the same results as that in Exercise 2.

For further thought: what happens if instead of throwing away the resources it collects the government uses all of its tax revenue to undertake investment?

**Exercise 4.** *Manna falls faster.*

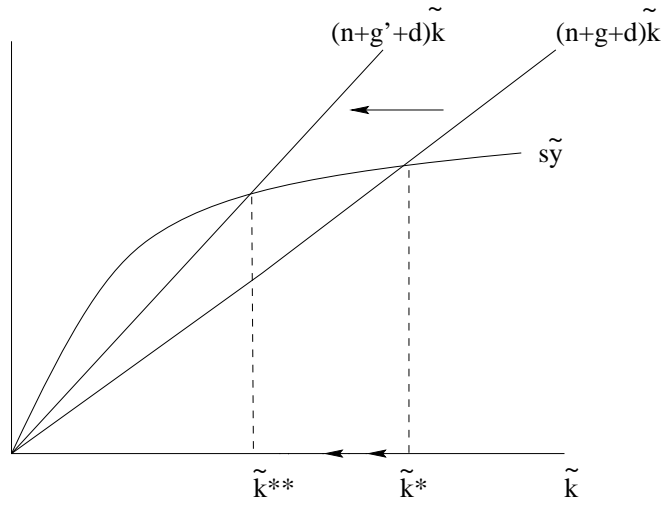
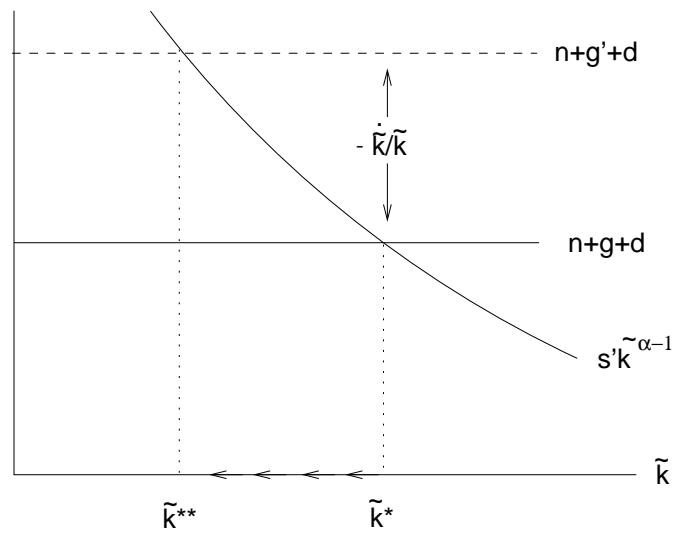
Figure 6 shows the Solow diagram for this question. It turns out, however, that it's easier to answer this question using the transition dynamics version of the diagram, as shown in Figure 7. When  $g$  rises to  $g'$ ,  $\dot{k}/k$  turns negative, as shown in Figure 7 and  $\dot{A}/A = g'$ , the new steady-state growth rate.

To see what this implies about the growth rate of  $y$ , recall that

$$\frac{\dot{y}}{y} = \frac{\dot{\tilde{y}}}{\tilde{y}} + \frac{\dot{A}}{A} = \alpha \frac{\dot{\tilde{k}}}{\tilde{k}} + g'.$$

So to determine what happens to the growth rate of  $y$  at the moment of the change in  $g$ , we have to determine what happens to  $\dot{\tilde{k}}/\tilde{k}$  at that moment. As can be seen in Figure 7, or by algebra, this growth rate falls to  $g - g' < 0$  — it is the negative of the difference between the two horizontal lines.

Substituting into the equation above, we see that  $\dot{y}/y$  immediately after the

Figure 6: An Increase in  $g$ Figure 7: An Increase in  $g$ : Transition Dynamics

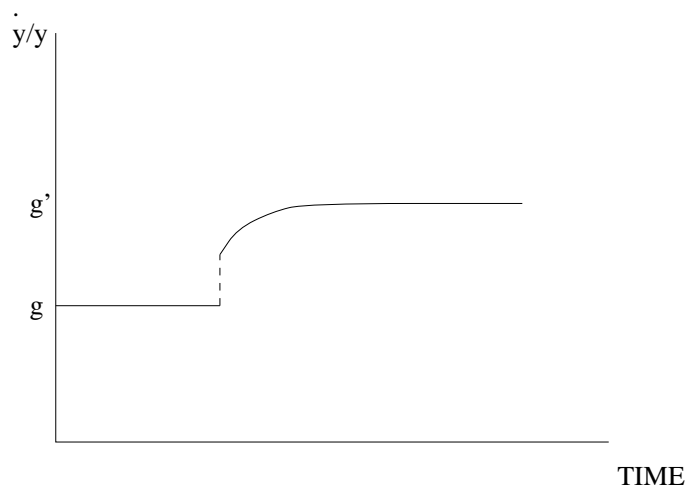
increase in  $g$  (suppose this occurs at time  $t = 0$ ) is given by

$$\frac{\dot{y}}{y} \Big|_{t=0} = \alpha(g - g') + g' = (1 - \alpha)g' + \alpha g > 0.$$

Notice that this value, which is a weighted average of  $g'$  and  $g$ , is strictly less than  $g'$ .

After time  $t = 0$ ,  $\dot{y}/y$  rises up to  $g'$  (which can be seen by looking at the dynamics implied by Figure 6). Therefore, we know that the dynamics of the growth rate of output per worker look like those shown in Figure 8.

Figure 8: Growth Rate of Output per Worker



**Exercise 5.** *Can we save too much?*

From the standard Solow model, we know that steady-state output per capita is given by  $y^* = \left(\frac{s}{n+d}\right)^{\frac{\alpha}{1-\alpha}}$ . Steady-state consumption per worker is  $(1-s)y^*$ , or

$$c^* = (1-s) \left(\frac{s}{n+d}\right)^{\frac{\alpha}{1-\alpha}}.$$

From this expression, we see that an increase in the saving rate has two effects. First, it increases steady-state output per worker and therefore tends to increase consumption. Second, it reduces the amount of output that gets consumed.

To maximize  $c^*$ , we take the derivative of this expression with respect to  $s$  and set it equal to zero:

$$\frac{\partial c^*}{\partial s} = - \left(\frac{s}{n+d}\right)^{\frac{\alpha}{1-\alpha}} + (1-s) \frac{\alpha}{1-\alpha} \frac{s^{\frac{\alpha}{1-\alpha}-1}}{(n+d)^{\frac{\alpha}{1-\alpha}}} = 0.$$

Rearrange the equation, we have

$$1 = \frac{1-s^*}{s^*} \frac{\alpha}{1-\alpha},$$

and therefore

$$s^* = \alpha.$$

The saving rate which maximizes the steady-state consumption equals  $\alpha$ .

Now turn to the marginal product of capital,  $MPK$ . Given the production function  $y = k^\alpha$ , the marginal product of capital is  $\alpha k^{\alpha-1}$ . Evaluated at the steady state value  $k^*$ ,

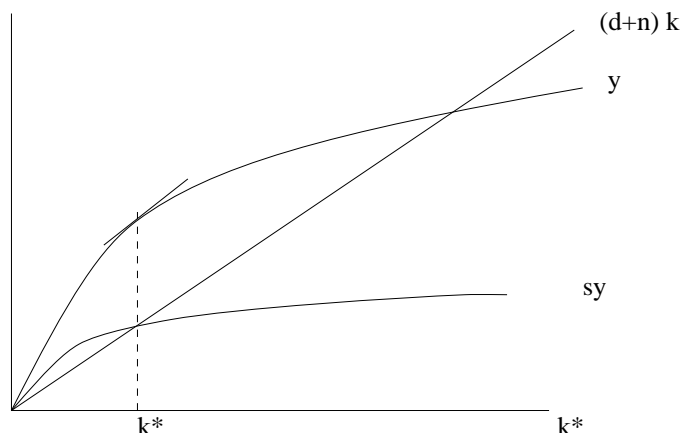
$$MPK = \alpha (k^*)^{\alpha-1} = \alpha \left(\frac{n+d}{s}\right).$$

When the saving rate is set to maximize consumption per person,  $s^* = \alpha$ , so that the marginal product of capital is

$$MPK^* = n + d.$$

That is, the steady-state marginal product of capital equals  $n + d$  when consumption per person is maximized. Alternatively, this expression suggests that the *net* marginal product of capital — i.e. the marginal product of capital net of depreciation — is equal to the population growth rate. This relationship is graphed in Figure 9.

Figure 9: Can We Save Too Much?



If  $s > \alpha$ , then steady-state consumption could be increased by reducing the saving rate. This result is related to the diminishing returns associated with capital accumulation. The higher is the saving rate, the lower is the marginal product of capital. The marginal product of capital is the return to investing — if you invest one unit of output, how much do you get back? The intuition is clearest if we set  $n = 0$  for the moment. Then, the condition says that the marginal product of capital should equal the rate of depreciation, or the net return to capital should be zero. If the marginal product of capital falls below the rate of depreciation, then you are getting back less than you put in, and therefore you are investing too much.

**Exercise 6.** *Solow (1956) versus Solow (1957).*

**a)** This is an easy one. Growth in output per worker in the initial steady state is 2 percent and in the new steady state is 3 percent.

**b)** Recall equation (2.15)

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + \frac{\dot{B}}{B}$$

	$\frac{\dot{y}}{y}$	$\alpha \frac{\dot{k}}{k}$	$\frac{\dot{B}}{B}$
Initial S.S.	.02	$1/3 * (.02)$	$2/3 * (.02) = .0133$
New S.S.	.03	$1/3 * (.03)$	$2/3 * (.03) = .0200$
Change	.01	$1/3 * (.01)$	$2/3 * (.01) = .0067$



In other words, Solow (1957) would say that  $1/3$  of the faster growth in output per worker is due to capital and  $2/3$  is due to technology.

c) The growth accounting above suggests attributing some of the faster growth to capital and some to technology. Of course this is true in an accounting sense. However, we know from Solow (1956) that faster growth in technology is itself the cause of the faster growth in capital per worker. It is in this sense that the accounting picture can sometimes be misleading.